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The nested $SU(N)$ off-shell Bethe ansatz and exact form factors

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Abstract
The form factor equations are solved for a $SU(N)$ invariant S-matrix under the assumption that the anti-particle is identified with the bound state of $N - 1$ particles. The solution is obtained explicitly in terms of the nested off-shell Bethe ansatz where the contribution from each level is written in terms of multiple contour integrals. The general solution is illustrated for some operators, such as the energy–momentum tensor, the fields and the current.

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1. Introduction

Thirty five years after the discovery of asymptotic freedom [1, 2], quantum chromodynamics (QCD), the theory of the strong interactions between quarks and gluons remains a challenge. One of the most important trends in theoretical physics in the last several decades is the development of exact methods which are completely different from the perturbation theory since the perturbation theory is not applicable in the strong coupling regime. Resolution of the strong coupling problem would give us a full understanding of the structure of interactions in non-Abelian gauge theory and, in particular, would shed light on the mysterious confinement phenomena in QCD. One promising possibility of overcoming the limitations of the perturbation theory is the application of exact integrability. This is one of the reasons why investigation of two-dimensional exactly solvable lattice statistical systems, one-dimensional spin chains and quantum field theory models in 1 + 1 dimensions in the framework of the Bethe ansatz method is still quite actual [3]. Some two-dimensional exactly integrable field theories, such as a nonlinear σ-model or Gross–Neveu model, exhibit asymptotic freedom, a phenomenon initially observed in non-Abelian gauge theories. From this point of view the two-dimensional integrable quantum field theories are in a sense a laboratory for the investigations of those properties of quantum field theories, which cannot be described via perturbation theory. One may hope that some of these properties, at least qualitatively, could be extended to the higher dimensions too.

The *Bethe ansatz method* [4], was first introduced by Bethe to solve the isotropic Heisenberg model. Yang [5] and Sutherland [6] generalized the technique of the Bethe ansatz for those cases where the underlying symmetry group is larger than $SU(2)$. This method is now called the *nested Bethe ansatz*. A great impulse in the theory of integrable models was given by Faddeev and collaborators [7] with the development of the algebraic Bethe ansatz method. Another modification of the Bethe ansatz is the *off-shell Bethe ansatz*, which was originally formulated in [8] to calculate the correlation function in WZNW models (see also [9, 10]). This version of the Bethe ansatz paves the way to an analysis of off-shell quantities and opens up the intriguing possibility of merging the Bethe ansatz and the form factor approach.

The exact determination of form factors, which are matrix elements of local operators, was formulated in the 1970s [11]. This approach was developed further and studied in the context of several explicit models by Smirnov [12] . Subsequently, [13] the techniques of the off-shell Bethe ansatz were used to determine the form factors for the sine-Gordon model. There, however, the underlying group structure is simple and there was no need to use a nested version of the off-shell Bethe ansatz. Recently, the form factor program has received renewed interest in different areas, such as condensed matter physics [14–16] and atomic physics [17]. In particular, applications to Mott insulators and carbon nanotubes [14, 18] doped two-leg ladders [19] and in the field of Bose–Einstein condensates of ultracold atomic and molecular gases [17, 20] have been discussed. In addition, form factors have also been employed to compute corrections to the entanglement, an essential property in the field of quantum computation [21].

In the present paper we will focus on the determination of the form factors for an $SU(N)$ invariant S-matrix. We have to apply the nested off-shell Bethe ansatz to get the vectorial part of the form factors. This solves the missing link of Smirnov's [12] formula for the $SU(N)$ form factors, where the vectors were given by an 'indirect definition' characterized by necessary properties but not provided explicitly. The procedure is similar as for the scaling $Z(N)$ ising and affine $A(N-1)$ Toda models [22, 23] because the bound-state structures of these models are similar. However, the algebraic structure of the form factors for the $SU(N)$ model is more complicated, because the S-matrix possesses backward scattering. Therefore, we have to develop a nontrivial algebraic *nested off-shell Bethe ansatz*. We note that $SU(N)$ form factors were also calculated in [12, 24, 25] using other techniques, also see the related paper [26, 27]. We believe that our integral representation may shed some light, for a better understanding, on the correlation functions of $SU(N)$ invariant models, which are also interesting for applications [28].

The paper is organized as follows: In section 2, we recall some known formulae and derive some simple results, which will be used in the following. In section 3, we construct the general form factor formula. In section 4, we discuss form factors of some operators such as the energy–momentum tensor, the Dirac field and the $SU(N)$ current. A summary of our results can be found in section 5. In appendix A we present some basic properties of the $SU(N)$ bound state S-matrix. In appendix B we provide the proofs of the main theorem, and in appendix C we prove the lemma dedicated to the higher level Bethe ansatz.

2. General setting

2.1. The $SU(N)$ S-matrix

The general solutions of the Yang–Baxter equations, unitarity and crossing relations for a $U(N)$ invariant *S*-matrix have been obtained in [29]. The *S*-matrix for the scattering of two particles of rank 1 (belonging to the vector representation of $SU(N)$) can be written as

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = \delta_{\alpha\gamma}\delta_{\beta\delta}b(\theta) + \delta_{\alpha\delta}\delta_{\beta\gamma}c(\theta). \tag{1}$$

The amplitudes which we use in the following are given by

$$a(\theta) = b(\theta) + c(\theta) = -\frac{\Gamma\left(1 - \frac{\theta}{2\pi \mathrm{i}}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi \mathrm{i}}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi \mathrm{i}}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi \mathrm{i}}\right)} \tag{2}$$

and

$$c(\theta) = -\frac{\mathrm{i}\eta}{\theta}b(\theta), \qquad \eta = 2\pi/N.$$

The amplitude $a(\theta)$ is the highest weight *S*-matrix eigenvalue for the two-particle scattering. It will be essential for the Bethe ansatz below.

The *S*-matrix eigenvalue $b(\theta) - c(\theta)$ has a pole at $\theta = \mathrm{i}\eta$, which means that there exist bound states of r fundamental particles $\alpha_1 + \cdots + \alpha_r \to (\rho_1 \cdots \rho_r)$ (with $\rho_1 < \cdots < \rho_r$) which transform as the anti-symmetric $SU(N)$ tensor representation of rank $r (0 < r < N)$. The masses of the bound states satisfy $m_r = m_{N-r}$ which suggests Swieca's [30, 31] picture that the anti-particle of a particle of rank r is to be identified with the particle of rank $N - r$ (also see [22, 23]).

The general bound state *S*-matrix formula [32, 33] for the scattering of a bound state with another particle can be used to calculate iteratively the scattering of a particle of rank $N - 1$ with a particle of rank 1 (for details, see appendix A and [34]). The result is

$$S_{(\rho)\alpha}^{\gamma(\sigma)}(\theta) = (-1)^{N-1}\left(\delta_{(\rho)}^{(\sigma)}\delta_\alpha^\gamma b(\pi\mathrm{i} - \theta) + \mathbf{C}^{\gamma(\sigma)}\mathbf{C}_{(\rho)\alpha}c(\pi\mathrm{i} - \theta)\right), \tag{3}$$

where $(\sigma) = (\sigma_1 \cdots \sigma_{N-1})$ and $(\rho) = (\rho_1 \cdots \rho_{N-1})$ denote the bound states. We have introduced the charge conjugation matrices

$$\begin{aligned} &\mathbf{C}_{(\alpha_1\cdots\alpha_{N-1})\alpha_N} = \mathbf{C}_{\alpha_1(\alpha_2\cdots\alpha_N)} = \epsilon_{\alpha_1\cdots\alpha_N} \\ &\mathbf{C}^{\alpha_1(\alpha_2\cdots\alpha_N)} = \mathbf{C}^{(\alpha_1\cdots\alpha_{N-1})\alpha_N} = (-1)^{N-1}\epsilon^{\alpha_1\alpha_2\cdots\alpha_N} \end{aligned} \tag{4}$$

with $\epsilon_{\alpha_1\cdots\alpha_N}$ and $\epsilon^{\alpha_1\alpha_2\cdots\alpha_N}$ being total anti-symmetric and $\epsilon_{1\dots N} = \epsilon^{1\cdots N} = 1$. The charge conjugation matrices satisfy

$$\mathbf{C}_{\alpha(\rho)}\mathbf{C}^{(\rho)\beta} = \delta_\alpha^\beta, \qquad \mathbf{C}_{\alpha(\rho)}A_\beta^\alpha\mathbf{C}^{\beta(\rho)} = (-1)^{N-1}\operatorname{tr} A. \tag{5}$$

Formula (3) may be interpreted as an unusual (because of the sign $(-1)^{N-1}$) crossing relation:

$$S_{(\rho)\alpha}^{\gamma(\sigma)}(\theta) = (-1)^{N-1}\mathbf{C}_{(\rho)\delta}S_{\alpha\beta}^{\delta\gamma}(\pi\mathrm{i} - \theta)\mathbf{C}^{\beta(\sigma)}$$

supporting Swieca's picture that the anti-particle of a particle of rank 1 is to be identified with the bound state of rank $N - 1$.

In order to simplify the formulae, we extract the factor $a(\theta)$ from the *S*-matrix (1).

$$S_{\alpha\beta}^{\delta\gamma}(\theta) = a(\theta)\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta),$$

such that

$$\begin{aligned} &\tilde{S}_{\alpha\beta}^{\delta\gamma}(\theta) = \delta_\alpha^\gamma\delta_\beta^\delta\tilde{b}(\theta) + \delta_\alpha^\delta\delta_\beta^\gamma\tilde{c}(\theta) \\ &\tilde{b}(\theta) = \frac{\theta}{\theta - \mathrm{i}\eta}, \qquad \tilde{c}(\theta) = \frac{-\mathrm{i}\eta}{\theta - \mathrm{i}\eta}. \end{aligned} \tag{6}$$

2.2. Generalized form factors

For a state of n particles of kind α_i with rapidities θ_i and a local operator $\mathcal{O}(x)$ we define the form factor functions $F^{\mathcal{O}}_{\alpha_1\cdots\alpha_n}(\theta_1,\ldots,\theta_n)$, or using a short-hand notation $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$, by

$$\langle 0|\mathcal{O}(x)|\theta_1,\ldots,\theta_n\rangle^{\text{in}}_{\underline{\alpha}} = \mathrm{e}^{-\mathrm{i}x(p_1+\cdots+p_n)} F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}), \qquad \text{for} \quad \theta_1 > \cdots > \theta_n, \tag{7}$$

where $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\underline{\theta} = (\theta_1,\ldots,\theta_n)$. For all other arrangements of the rapidities the functions $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ are given by analytic continuation. Note that the physical value of the form factor, i.e. the left-hand side of (7), is given for ordered rapidities as indicated above and the statistics of the particles. $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ are considered as the components of a co-vector-valued function $F^{\mathcal{O}}_{1\cdots n}(\underline{\theta}) \in V_{1\cdots n} = (V^{1\cdots n})^\dagger$. The general form factor can be written as [11]

$$F^{\mathcal{O}}_{1\cdots n}(\underline{\theta}) = K^{\mathcal{O}}_{1\cdots n}(\underline{\theta}) \prod_{1\leqslant i<j\leqslant n} F(\theta_{ij}) \tag{8}$$

where $F(\theta)$ is the minimal form factor function (9) and the K-function $K^{\mathcal{O}}_{1\cdots n}(\underline{\theta})$ is given below. It satisfies the form factor equations (i)–(v) (see below and [11–13, 23, 33]). For bosons and fermions they follow from general LSZ assumptions and 'maximal analyticity' [13]. For anyons there are additional statistics factors [23, 35, 36]. As we will discuss in section 3, the statistics factors are modified due to the unusual crossing relation of the S-matrix (see [31]). We will also provide a constructive and systematic way of how to solve the form factor equations (i)–(v) for the co-vector-valued function $F^{\mathcal{O}}_{1\cdots n}$ for the $SU(N)$ S-matrix.

Minimal form factor. The solutions of Watson's and the crossing equations (i) and (ii) for two particles with no poles in the physical strip $0 \leqslant \operatorname{Im}\theta \leqslant \pi$ and at most a simple zero at $\theta = 0$ are the minimal form factors. In particular, those for highest weight states are essential for the construction of the off-shell Bethe ansatz. One easily finds the minimal solution of

$$F(\theta) = a(\theta)F(-\theta) = F(2\pi\mathrm{i} - \theta)$$

using (2) as

$$F(\theta) = c\exp\int_0^\infty \frac{\mathrm{d}t}{t\sinh^2 t}\,\mathrm{e}^{\frac{t}{N}}\sinh t(1-1/N)(1-\cosh t(1-\theta/(\mathrm{i}\pi))). \tag{9}$$

It belongs to the highest weight vector $w = (2,0,\ldots,0)$ of the two-particle state. We define the corresponding 'Jost function' as for the $Z(N)$ models [22, 23] by the equation

$$\prod_{k=0}^{N-2}\phi(\theta + k\mathrm{i}\eta)\prod_{k=0}^{N-1}F(\theta + k\mathrm{i}\eta) = 1, \qquad \eta = 2\pi/N, \tag{10}$$

which is typical for models where the bound state of $N-1$ particles is the anti-particle [23]. The solution is

$$\phi(\theta) = \Gamma(\theta/(2\pi\mathrm{i}))\Gamma(1 - 1/N - \theta/(2\pi\mathrm{i})), \tag{11}$$

and satisfies the relations

$$\begin{aligned}\phi(\theta) &= \phi(-\theta)a(-\theta) = \phi((N-1)\mathrm{i}\eta - \theta)\\ &= \frac{1}{-b(\theta)}\phi(2\pi\mathrm{i} - \theta) = \frac{a(\theta - 2\pi\mathrm{i})}{-b(\theta)}\phi(\theta - 2\pi\mathrm{i}).\end{aligned} \tag{12}$$

Equations (10) and (11) also determine the normalization constant c in (9) (for details see appendix A and [34]).

2.3. Nested 'off-shell' Bethe ansatz for $SU(N)$

We consider a state with n particles and define as usual in the context of the algebraic Bethe ansatz [7, 37] the monodromy matrix

$$\tilde{T}_{1\cdots n,0}(\underline{\theta}, \theta_0) = \tilde{S}_{10}(\theta_{10}) \cdots \tilde{S}_{n0}(\theta_{n0}). \tag{13}$$

It is a matrix acting in the tensor product of the 'quantum space' $V^{1\cdots n} = V^1 \otimes \cdots \otimes V^n$ and the 'auxiliary space' V^0. All vector spaces V^i are isomorphic to a space V whose basis vectors label all kinds of particles. Here we consider $V \cong \mathbb{C}^N$ as the space of the vector representation of $SU(N)$. The Yang–Baxter algebra relation for the S-matrix yields

$$\tilde{T}_{1\cdots n,a}(\underline{\theta}, \theta_a)\tilde{T}_{1\cdots n,b}(\underline{\theta}, \theta_b)\tilde{S}_{ab}(\theta_a - \theta_b) = \tilde{S}_{ab}(\theta_a - \theta_b)\tilde{T}_{1\cdots n,b}(\underline{\theta}, \theta_b)\tilde{T}_{1\cdots n,a}(\underline{\theta}, \theta_a),$$

which implies the basic algebraic properties of the sub-matrices A, B, C, D with respect to the auxiliary space defined by

$$\tilde{T}_{1\cdots n,0}(\underline{\theta}, z) \equiv \begin{pmatrix} \tilde{A}_{1\cdots n}(\underline{\theta}, z) & \tilde{B}_{1\cdots n,\beta}(\underline{\theta}, z) \\ \tilde{C}^{\beta}_{1\cdots n}(\underline{\theta}, z) & \tilde{D}^{\beta'}_{1\cdots n,\beta}(\underline{\theta}, z) \end{pmatrix}, \tag{14}$$

where $2 \leqslant \beta, \beta' \leqslant N$.

An 'off-shell' Bethe ansatz for a co-vector-valued function $K_{\underline{\alpha}}(\underline{\theta})$ is given by the multiple contour integral

$$\boxed{K_{\underline{\alpha}}(\underline{\theta}) = \frac{1}{m!}\int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_1}{R} \cdots \int_{\mathcal{C}_{\underline{\theta}}} \frac{dz_m}{R} \tilde{h}(\underline{\theta}, \underline{z}) p(\underline{\theta}, \underline{z}) \tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z})} \tag{15}$$

with the scalar functions

$$\tilde{h}(\underline{\theta}, \underline{z}) = \prod_{i=1}^{n}\prod_{j=1}^{m} \tilde{\phi}(\theta_i - z_j) \prod_{1\leqslant i<j\leqslant m} \tau(z_i - z_j) \tag{16}$$

$$\tau(z) = \frac{1}{\phi(\theta)\phi(-\theta)}, \qquad \tilde{\phi}(\theta) = \phi(\theta)a(\theta) = \phi(-\theta), \tag{17}$$

where the function $\phi(\theta)$ is defined by equation (10) with solution (11). Below we will use the co-vector-valued function $K_{\underline{\alpha}}(\underline{\theta})$ to construct form factors for operators $\mathcal{O}(x)$. The scalar p-function $p(\underline{\theta}, \underline{z})$ which is in general a simple function of e^{θ_i} and e^{z_j} (see below) will depend on the specific operator $\mathcal{O}(x)$. It turns out that by means of the integral representation, we transform the complicated form factor equations (22)–(26) (which are in general matrix equations) to simple equations for the p-function (see section 3). The integration contour $\mathcal{C}_{\underline{\theta}}$ (see figure 1) can be characterized as follows: there is a finite number of complex numbers $a_i(\underline{\theta}), b_j(\underline{\theta})$ such that the positions of all poles of the integrand in (15) are of the form

$$(1) \quad a_i(\underline{\theta}) + 2\pi i k, \quad k \in \mathbb{N}, \qquad (2) \quad b_j(\underline{\theta}) - 2\pi i l, \quad l \in \mathbb{N},$$

then $\mathcal{C}_{\underline{\theta}}$ runs from $-\infty$ to $+\infty$ such that all poles (1) are above and all poles (2) are below the contour (this is usually not a straight line). This contour is just the same as that used for the definition of Meijer's G-function. The constant R is defined by $R = \oint_\theta dz \tilde{\phi}(\theta - z)$ where the integration contour is a small circle around $z = \theta$ as part of $\mathcal{C}_{\underline{\theta}}$. The K-function is in general a linear combination of the *fundamental building blocks* [22, 23, 38] given by (15)–(17). We consider here cases where the sum consists only of one term.

Nesting. If in (14) the range of β's is nontrivial, i.e. if $N > 2$, the Bethe ansatz co-vectors are of the form

$$\tilde{\Psi}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) = L_{\underline{\beta}}(\underline{z})\tilde{\Phi}^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) \tag{18}$$

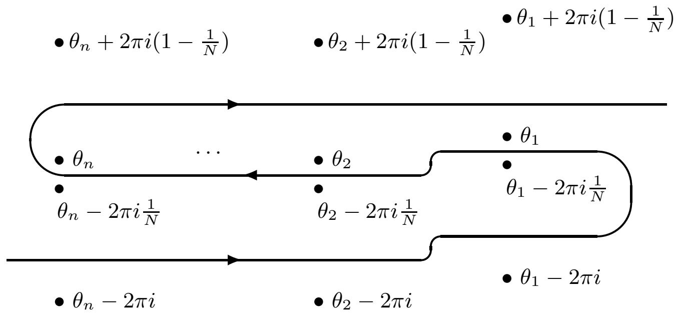

Figure 1. The integration contour $\mathcal{C}_{\underline{\theta}}$. The bullets refer to poles of the integrand in (15).

where summation over all $\underline{\beta} = (\beta_1, \ldots, \beta_m)$ with $\beta_i > 1$ is assumed. The basic Bethe ansatz co-vectors $\tilde{\Phi}^{\underline{\beta}}_{1\cdots n} \in (V^{1\cdots n})^{\dagger}$ are defined as

$$\tilde{\Phi}^{\underline{\beta}}_{1\cdots n}(\underline{\theta}, \underline{z}) = \Omega_{1\cdots n}\tilde{C}^{\beta_m}_{1\cdots n}(\underline{\theta}, z_m)\cdots\tilde{C}^{\beta_1}_{1\cdots n}(\underline{\theta}, z_1). \tag{19}$$

Here the 'pseudo-vacuum' is the highest weight co-vector (with weight vector $w = (n, 0, \ldots, 0)$)

$$\Omega_{1\cdots n} = e(1) \otimes \cdots \otimes e(1)$$

where the unit vectors $e(\alpha)(\alpha = 1, \ldots, N)$ correspond to the particle of type α which belongs to the vector representation of $SU(N)$. The pseudo-vacuum vector satisfies

$$\begin{aligned} &\Omega_{1\cdots n}\tilde{B}^{\beta}_{1\cdots n}(\underline{\theta}, z) = 0, \\ &\Omega_{1\cdots n}\tilde{A}_{1\cdots n}(\underline{\theta}, z) = \Omega_{1\cdots n}, \\ &\Omega_{1\cdots n}\tilde{D}^{\beta'}_{1\cdots n,\beta}(\underline{\theta}, z) = \delta^{\beta'}_{\beta}\prod_{i=1}^{n}\tilde{b}(\theta_i - z)\Omega_{1\cdots n}. \end{aligned} \tag{20}$$

The amplitudes of the scattering matrices are given by equations (1) and (2). The technique of the 'nested Bethe ansatz' means that one makes for the coefficients $L_{\underline{\beta}}(\underline{z})$ in (18) the analogous construction (15)–(17) as for $K_{\underline{\alpha}}(\underline{\theta})$ where now the indices $\underline{\beta}$ take only the values $2 \leqslant \beta_i \leqslant N$. This nesting is repeated until the space of the coefficients becomes one dimensional. It is well known (see [39]) that the 'off-shell' Bethe ansatz states are highest weight states if they satisfy a matrix difference equation which is here the form factor equation (ii) (see below). The $SU(N)$ weights are

$$w = (n_0 - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1}) \tag{21}$$

where $n_0 = n, n_1 = m, \ldots$ are the numbers of variables in the various levels of the nesting.

3. $SU(N)$ form factors

In this section we perform the form factor program for the general $SU(N)$ S-matrix. For this purpose, we perform the nested Bethe ansatz method with $N - 1$ levels combined with the off-shell Bethe ansatz. Applications of the results to the $SU(N)$ chiral Gross–Neveu model [40] will be investigated in a separate article [34]. We consider operators $\mathcal{O}(x)$ with charge $Q^{\mathcal{O}}$ where $Q^{\mathcal{O}}$ is the smallest number n of fundamental (rank 1) particles such that

$\langle 0|\mathcal{O}(x)|\theta_1,\ldots,\theta_n\rangle^{\text{in}}$ is non-vanishing. For example, for the energy–momentum tensor and the $SU(N)$ current $Q^T = Q^J = 0$ and for the fundamental field $Q^\psi = 1$.

$SU(N)$ form factor equations. The co-vector-valued auxiliary function $F^{\mathcal{O}}_{1\cdots n}(\underline{\theta})$ is meromorphic in all variables $\theta_1,\ldots,\theta_n$ and satisfies the following relations.

(i) The Watson's equations describe the symmetry property under the permutation of both, the variables θ_i,θ_j and the spaces $i, j = i+1$ at the same time,

$$F^{\mathcal{O}}_{\cdots ij\cdots}(\cdots,\theta_i,\theta_j,\ldots) = F^{\mathcal{O}}_{\cdots ji\cdots}(\cdots,\theta_j,\theta_i,\ldots)S_{ij}(\theta_{ij}) \tag{22}$$

for all possible arrangements of θ's.

(ii) The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces:

$$\begin{aligned} {}^{\text{out},\bar{1}}\langle\theta_1|\mathcal{O}(0)|\theta_2,\ldots,\theta_n\rangle^{\text{in,conn.}}_{2\cdots n} &= F^{\mathcal{O}}_{1\ldots n}(\theta_1+\mathrm{i}\pi,\theta_2,\ldots,\theta_n)\dot{\sigma}_1\mathbf{C}^{\bar{1}1} \\ &= F^{\mathcal{O}}_{2\ldots n1}(\theta_2,\ldots,\theta_n,\theta_1-\mathrm{i}\pi)\mathbf{C}^{1\bar{1}} \end{aligned} \tag{23}$$

where $\dot{\sigma}^{\mathcal{O}}_\alpha$ takes into account the statistics of the particle α with respect to $\mathcal{O}$. The charge conjugation matrix $\mathbf{C}^{\bar{1}1}$ is defined by (4).

(iii) There are poles determined by one-particle states in each sub-channel given by a subset of particles of the state in (7). In particular, the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = \mathrm{i}\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\pi} F^{\mathcal{O}}_{1\cdots n}(\theta_1,\ldots,\theta_n) = 2\mathrm{i}\mathbf{C}_{12}F^{\mathcal{O}}_{3\cdots n}(\theta_3,\ldots,\theta_n)(\mathbf{1}-\dot{\sigma}_2S_{2n}\cdots S_{23}). \tag{24}$$

(iv) If there are also bound states in the model the function $F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If, for instance, particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = \mathrm{i}\eta$, $(0<\eta<\pi)$ such that

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\eta} F^{\mathcal{O}}_{12\cdots n}(\theta_1,\theta_2,\ldots,\theta_n) = F^{\mathcal{O}}_{(12)\cdots n}(\theta_{(12)},\ldots,\theta_n)\sqrt{2}\Gamma^{(12)}_{12} \tag{25}$$

where the bound-state intertwiner $\Gamma^{(12)}_{12}$ and the values of $\theta_1,\theta_2,\theta_{(12)}$ and η are given in [32, 33].

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F^{\mathcal{O}}_{1\cdots n}(\theta_1,\ldots,\theta_n) = \mathrm{e}^{s\mu}F^{\mathcal{O}}_{1\cdots n}(\theta_1+\mu,\ldots,\theta_n+\mu), \tag{26}$$

if the local operator transforms under Lorentz transformations as $\mathcal{O}\to \mathrm{e}^{s\mu}\mathcal{O}$ where s is the 'spin' of $\mathcal{O}$.

Here $\dot{\sigma}_1 = \rho\sigma^{\mathcal{O}}_1$ is a phase factor, where $\sigma^{\mathcal{O}}_1$ is the statistics factor of the operator $\mathcal{O}(x)$ with respect to particle 1, and ρ is a sign factor due to the unusual crossing relation of the S-matrix which is determined as follows. The statistics factors in (ii) and (iii) are not arbitrary, consistency implies that both are the same, and

$$\dot{\sigma}^{\mathcal{O}}_1\dot{\sigma}^{\mathcal{O}}_{\bar{1}} = (\dot{\sigma}^{\mathcal{O}}_1)^N = (-1)^{(N-1)Q^{\mathcal{O}}}$$

with the solution

$$\sigma^{\mathcal{O}}_1 = \mathrm{e}^{\mathrm{i}\pi(1-1/N)Q^{\mathcal{O}}} \tag{27}$$

$$\rho = (-1)^{(N-1)+(1-1/N)(n-Q^{\mathcal{O}})} = \pm 1 \tag{28}$$

where $Q^{\mathcal{O}} = n\,\mathrm{mod}N$ is the charge of the operator $\mathcal{O}(x)$. This solution and the decomposition of $\dot{\sigma}^{\mathcal{O}}_1 = \rho\sigma^{\mathcal{O}}_1$ are of course not unique; the choice proposed is consistent with [31] and the usual relation of spin and statistics (for all examples discussed below).

3.1. The form factor formula and $SU(N)$ weights

In order to obtain a recursion relation where only form factors for the fundamental particles of type α (which transform as the $SU(N)$ vector representation) are involved, we have to apply iteratively the bound-state relation (iv) to get the $(N-1)$ bound state which is to be identified with the anti-particle. Using then the annihilation residue equation (iii), we obtain with (5)

$$\operatorname*{Res}_{\theta_{N-1N}=\mathrm{i}\eta}\cdots\operatorname*{Res}_{\theta_{12}=\mathrm{i}\eta} F^{\mathcal{O}}_{123\cdots n}(\theta_1,\ldots,\theta_n)$$
$$= 2\mathrm{i}\sqrt{2}^{N-2}\Gamma\varepsilon_{1\ldots N} F^{\mathcal{O}}_{N+1\cdots n}(\theta_{N+1},\ldots,\theta_n)(\mathbf{1}-\dot\sigma^{\mathcal{O}}_N S_{Nn}\cdots S_{NN+1}) \tag{29}$$

where Γ is a constant (see (A.5) in appendix A).

The properties of the K-function: We proposed above the n-particle form factors of an operator $\mathcal{O}(x)$ as given by formula (8). In this expression, the K-function $K^{\mathcal{O}}_{1\cdots n}(\underline{\theta})$ contains the entire pole structure and its properties are determined by the form factor equations (i)–(iii) which read in terms of the K-function as

$$K_{\ldots ij\ldots}(\cdots,\theta_i,\theta_j,\ldots) = K_{\ldots ji\ldots}(\cdots,\theta_j,\theta_i,\ldots)\tilde S_{ij}(\theta_{ij}) \tag{30}$$

$$K_{\alpha_1\alpha_2\ldots\alpha_n}(\theta_1+\mathrm{i}\pi,\theta_2,\ldots,\theta_n)\dot\sigma_{\alpha_1}(-1)^{N-1} = K_{\alpha_2\ldots\alpha_n\alpha_1}(\theta_2,\ldots,\theta_n,\theta_1-\mathrm{i}\pi) \tag{31}$$

$$\operatorname*{Res}_{\theta_{N-1N}=\mathrm{i}\eta}\cdots\operatorname*{Res}_{\theta_{12}=\mathrm{i}\eta} K_{1\cdots N}(\underline{\theta}) = c_0\prod_{i=N+1}^{n}\prod_{j=2}^{N}\tilde\phi(\theta_{ij})\varepsilon_{1\cdots N}$$
$$\times K_{N+1\cdots n}(\theta_{N+1},\ldots,\theta_n)(\mathbf{1}-\dot\sigma_N S_{Nn}\cdots S_{NN+1}) \tag{32}$$

with the constant

$$c_0 = 2\mathrm{i}\sqrt{2}^{N-2}\Gamma\prod_{j=1}^{N-1}F^{-(N-j)}(j\mathrm{i}\eta). \tag{33}$$

For the second equation (31), relation (5) for the charge conjugation matrices has been used and the third equation (32) follows from (10) and (29).

We propose the K-function $K_{\underline{\alpha}}(\underline{\theta})$ as the integral representation (15) with (16) and (17). The Bethe ansatz co-vector $\tilde\Psi_{\underline{\alpha}}(\underline{\theta},\underline{z})$ is of the forms (18) and (19), and for the function $L_{\underline{\beta}}(\underline{z})$ one makes again an analogous ansatz as for the K-function (15) where the indices run over sets with one element less. For the $SU(N)$ case, we have to iterate this $N-2$ times.

The higher level Bethe ansatz. We define for $l=1,\ldots,N-2$, with the short notation $m=n_l$ and $k=n_{l+1}$:

$$L^{(l)}_{\underline{\beta}}(\underline{z}) = \frac{1}{k!}\int_{\mathcal{C}_{\underline{z}}}\frac{\mathrm{d}u_1}{R}\cdots\int_{\mathcal{C}_{\underline{z}}}\frac{\mathrm{d}u_k}{R}\tilde h(\underline{z},\underline{u})p^{(l)}(\underline{z},\underline{u})\tilde\Psi^{(l)}_{\underline{\beta}}(\underline{z},\underline{u}) \tag{34}$$

$$\tilde h(\underline{z},\underline{u}) = \prod_{i=1}^{m}\prod_{j=1}^{k}\tilde\phi(z_i-u_j)\prod_{1\leqslant i<j\leqslant k}\tau(u_i-u_j)$$
$$\tilde\Psi^{(l)}_{\underline{\beta}}(\underline{z},\underline{u}) = L^{(l+1)}_{\underline{\gamma}}(\underline{u})\tilde\Phi^{(l-1)\underline{\gamma}}_{\underline{\beta}}(\underline{z},\underline{u}) \tag{35}$$

where $\underline{z}=z_1,\ldots,z_m$, $\underline{u}=u_1,\ldots,u_k$, $\underline{\beta}=\beta_1,\ldots,\beta_m$, $\underline{\gamma}=\gamma_1,\ldots,\gamma_k$. The indices assume the values $\beta_i=l+1,\ldots,N$, $\gamma_i=l+2,\ldots,N$. By this procedure, one obtains the nested Bethe ansatz[4].

[4] In [39] the nested off-shell Bethe ansatz was formulated in terms of 'Jackson-type integrals' instead of contour integrals.

In order that the form factor $F^{\mathcal{O}}(\underline{\theta})$ given by (8), (15)–(19) and (34) satisfies the form factor equations (i)–(iii), the higher level L-functions $L^{(l)}_{\underline{\beta}}(\underline{z})$ have to satisfy

(i)$^{(l)}$ Watson's equations

$$L^{(l)}_{\dots ij\dots}(\cdots, z_i, z_j, \dots) = L^{(l)}_{\dots ji\dots}(\cdots, z_j, z_i, \dots)\tilde{S}_{ij}(z_{ij}), \tag{36}$$

(ii)$^{(l)}$ the crossing relation

$$L^{(l)}_{\beta_1,\beta_2,\dots,\beta_m}(z_1+2\pi\mathrm{i}, z_2, \dots, z_m) = L^{(l)}_{\beta_2,\dots,\beta_m,\beta_1}(z_2, \dots, z_m, z_1) \tag{37}$$

and

(iii)$^{(l)}$ the function $L^{(l)}_{\underline{\beta}}(\underline{z})$ has to possess simple poles at $z_{12}, \dots, z_{N-l-1,N-l} = \mathrm{i}\eta$ and it has to factorize in the neighborhood of these poles as

$$L^{(l)}_{\underline{\beta}}(\underline{z}) \approx c_l \prod_{i=N-l+1}^{m} \prod_{j=2}^{N-l} \tilde{\phi}(z_{ij}) \tilde{S}^{N\cdots l+1}_{\hat{\underline{\beta}}}(\hat{\underline{z}}) L^{(l)}_{\check{\underline{\beta}}}(\check{\underline{z}}) \tag{38}$$

where we have used the short notations

$$\begin{aligned} \hat{\underline{\beta}} &= (\beta_1, \dots, \beta_{N-l}), & \check{\underline{\beta}} &= (\beta_{N-l+1}, \dots, \beta_m) \\ \hat{\underline{z}} &= (z_1, \dots, z_{N-l}), & \check{\underline{z}} &= (z_{N-l+1}, \dots, z_m). \end{aligned} \tag{39}$$

The S-matrix $\tilde{S}^{N\cdots l+1}_{\hat{\underline{\beta}}}(\hat{\underline{z}})$ describes the total scattering of the $N-l$ particles with rapidities $\hat{\underline{z}}$ for the initial and the final configurations of quantum numbers $\hat{\underline{\beta}}$ and $(N, \dots, l+1)$, respectively.

Theorem 1. *We make the following assumptions:*

(1) the p-function $p(\underline{\theta}, \underline{z})$ satisfies the equations

$$\left.\begin{array}{lll} (\mathrm{i}') & p_{nm}(\underline{\theta}, \underline{z}) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j \text{ and } z_i \leftrightarrow z_j \\ (\mathrm{ii}'_1) & p_{nm}(\underline{\theta}, \underline{z}) = \dot{\sigma}(-1)^{N-1+m} p_{nm}(\theta_1+2\pi\mathrm{i}, \theta_2, \dots, \underline{z}) \\ (\mathrm{ii}'_2) & p_{nm}(\underline{\theta}, \underline{z}) = (-1)^n p_{nm}(\underline{\theta}, z_1+2\pi\mathrm{i}, z_2, \dots) \\ (\mathrm{iii}'_1) & p_{nm}(\underline{\theta}, \hat{\underline{z}}\check{\underline{z}}) = \nu p_{n-Nm-N+1}(\check{\underline{\theta}}, \check{\underline{z}}) \\ (\mathrm{iii}'_2) & p_{nm}(\underline{\theta}, \hat{\underline{z}}\check{\underline{z}}) = (-1)^{m+N-1}\dot{\sigma}\nu p_{n-Nm-N+1}(\check{\underline{\theta}}, \check{\underline{z}}) \end{array}\right\} \tag{40}$$

where in $(\mathrm{iii}'_1)\hat{\underline{z}} = (\theta_2, \dots, \theta_N)$ *and in* $(\mathrm{iii}'_2)\hat{\underline{z}} = (\theta_1, \dots, \theta_{N-1})$ *with* $\theta_{12} = \cdots = \theta_{N-1N} = \mathrm{i}\eta$. *In* (iii') *the short notations (39) and* $\check{\underline{\theta}} = (\theta_{N+1}, \dots, \theta_n)$ *are used;*

(2) the higher level function $L^{(1)}_{\underline{\beta}}(\underline{z})$ satisfies $(\mathrm{i})^{(1)}$–$(\mathrm{iii})^{(1)}$ *of (36)–(38) and*

(3) the normalization constant ν satisfies

$$\nu = \frac{2\mathrm{i}\sqrt{2}^{N-2}\Gamma}{(N-1)!(\mathrm{i}\eta)^{N-1}} \prod_{l=1}^{N-1} (-1)^{n_l+l} (\tilde{\phi}(l\mathrm{i}\eta)F(l\mathrm{i}\eta))^{l-N} \tag{41}$$

then the co-vector-valued function $F_{\underline{\alpha}}(\underline{\theta})$ given by the ansatz (8) and the integral representation (15) satisfies the form factor equations (i)–(iii) *in the form of (30)–(32).*

The proof of this theorem can be found in appendix B.

Lemma 2. *The higher level function $L^{(l)}_{\underline{\beta}}(\underline{z})$ satisfies* $(\mathrm{i})^{(l)}$–$(\mathrm{iii})^{(l)}$, *if the higher level p-function in the integral representation (34) satisfies*

$(\mathrm{i}')^{(l)}$ $\quad p^{(l)}(\underline{z}, \underline{u})$ *is symmetric under* $z_i \leftrightarrow z_j$

$$(\mathrm{ii}')^{(l)} \quad \begin{cases} p^{(l)}_{mk}(\underline{z}, \underline{u}) = (-1)^k p^{(l)}_{mk}(\underline{z}', \underline{u}) \\ \qquad\qquad\;\; = (-1)^m p^{(l)}_{mk}(\underline{z}, \underline{u}') \end{cases} \tag{42}$$

$(\mathrm{iii}')^{(l)}$ $\quad p^{(l)}_{mk}(\underline{z}, z_2, \dots z_{N-l}, \check{\underline{u}})|_{z_{12}=\cdots=z_{z_{N-l-1,N-l}}=\mathrm{i}\eta} = p^{(l)}_{\check{m}\check{k}}(\check{\underline{z}}, \check{\underline{u}})$

with $\check m = m - N + l$, $\check k = m - N + l - 1$ *and* $m = n_l$, $k = n_{l+1}$. *We use the short notation* $\underline{z}' = z_1 + 2\pi\mathrm{i}, z_2, \dots, z_m$ *and* $\underline{u}' = u_1 + 2\pi\mathrm{i}, u_2, \dots, u_k$. *In* $(\mathrm{iii}')^{(l)}$, *the short notations (39) and* $\underline{\check u} = (u_{N-l}, \dots, u_k)$ *are used.*

The constants satisfy the recursion relation

$$c_l = c_{l+1}(-1)^{n_{l+1}-N+l+1}\prod_{j=1}^{N-l-1}\tilde\phi(j\mathrm{i}\eta) \tag{43}$$

with the solution

$$c_1 = \tilde\phi^{N-2}(\mathrm{i}\eta)\tilde\phi^{N-3}(2\mathrm{i}\eta)\cdots\tilde\phi((N-2)\mathrm{i}\eta)\prod_{l=1}^{N-2}(-1)^{n_{l+1}-N+l+1} \tag{44}$$

if $c_{N-1} = 1$.

This lemma is proved in appendix C.

A compact form for full form factors. We may write the above results as

$$K_{\underline\alpha}(\underline\theta) = \frac{1}{\underline{n}!}\int \underline{\underline{\mathrm{d}z}}\,\tilde h(\underline\theta, \underline{\underline z})\,p(\underline\theta, \underline{\underline z})\tilde\Psi_{\underline\alpha}(\underline\theta, \underline{\underline z}), \tag{45}$$

$$\tilde h(\underline\theta, \underline{\underline z}) = \tilde h(\underline\theta, \underline z^{(1)})\cdots\tilde h(\underline z^{(N-2)}, \underline z^{(N-1)}), \tag{46}$$

$$\tilde\Psi_{\underline\alpha}(\underline\theta, \underline{\underline z}) = \tilde\Phi_{\underline\beta^{(N-1)}}(\underline z^{(N-2)}, \underline z^{(N-1)})\cdots\tilde\Phi_{\underline\alpha}^{\underline\beta^{(1)}}(\underline\theta, \underline z^{(1)}), \tag{47}$$

with the short notations

$$\int \underline{\underline{\mathrm{d}z}} = \int \underline{\mathrm{d}z}^{(1)}\cdots\int \underline{\mathrm{d}z}^{(N-1)};\ \int \underline{\mathrm{d}z}^{(l)} = \int_{\mathcal{C}_{\underline z^{(l-1)}}}\frac{\mathrm{d}z_1^{(l)}}{R}\cdots\int_{\mathcal{C}_{\underline z^{(l-1)}}}\frac{\mathrm{d}z_{n_l}^{(l)}}{R}$$

$$\underline{n}! = n_1!\cdots n_{N-1}!;\ \underline{\underline z} = \underline z^{(1)}, \dots, \underline z^{(N-1)};\ \underline z^{(l)} = z_1^{(l)}, \dots, z_{n_l}^{(l)}.$$

The function $\tilde h(\underline\theta, \underline z)$ is given by (16) and (17).

The p-function and $SU(N)$ *weights.* In general the function $p(\underline\theta, \underline{\underline z})$ depends on the rapidities $\underline\theta$ and all integration variables $\underline z^{(l)} l = 1, \dots, N-1$. In order that the form factor $F^{\mathcal O}(\theta)$ given by (8) and (45) satisfies the form factor equations (i)–(iii), the p-function $p^{\mathcal O}(\underline\theta, \underline{\underline z})$ in (45) which depends on the explicit nature of the local operator $\mathcal{O}(x)$ is assumed to satisfy

$$\left.\begin{array}{ll}
(\mathrm{i}') & p(\underline\theta, \underline{\underline z}) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j \text{ and } z_i^{(l)} \leftrightarrow z_j^{(l)} \\
(\mathrm{ii}'_1) & p(\underline\theta, \underline{\underline z}) = \dot\sigma(-1)^{n_1+N-1}p(\theta_1 + 2\pi\mathrm{i}, \theta_2, \dots, \underline{\underline z}) \\
(\mathrm{ii}'_2) & p(\underline\theta, \underline{\underline z}) = (-1)^{w_l^{\mathcal O}+w_{l+1}^{\mathcal O}}p(\underline\theta, \dots, z_i^{(l)} + 2\pi\mathrm{i}, \dots), l = 1, \dots, N-1 \\
(\mathrm{iii}'_1) & p(\underline\theta, \underline{\hat z}^{(1)}\underline{\check z}^{(1)}, \dots) = \nu p(\underline{\check\theta}, \underline{\check z}^{(1)}, \dots) \\
(\mathrm{iii}'_2) & p(\underline\theta, \underline{\hat z}^{(1)}\underline{\check z}^{(1)}, \dots) = (-1)^{n_1+N-1}\dot\sigma\nu p(\underline{\check\theta}, \underline{\check z}^{(1)}, \dots) \\
(\mathrm{iii}'_3) & p(\underline\theta, \dots, \underline z^{(l)}, \underline{\hat z}^{(l+1)}\underline{\check z}^{(l+1)}, \dots) = p(\underline\theta, \dots, \underline z^{(l)}, \underline{\check z}^{(l+1)}, \dots) \\
(\mathrm{v}') & p(\underline\theta, \underline{\underline z}) = \mathrm{e}^{s\mu}p(\underline\theta + \mu, \underline{\underline z} + \mu)
\end{array}\right\} \tag{48}$$

where in (iii'_1): $\underline{\hat z}^{(1)} = (\theta_2, \dots, \theta_N)$ in (iii'_2): $\underline{\hat z}^{(1)} = (\theta_1, \dots, \theta_{N-1})$ and in (iii'_3): $\underline{\hat z}^{(l+1)} = (z_2^{(l)}, \dots z_{N-l}^{(l)})$, furthermore in (iii'_1) and (iii'_2): $\theta_{12} = \cdots = \theta_{N-1N} = \mathrm{i}\eta$ and in (iii'_3): $z_{12}^{(l)} = \cdots = z_{z_{N-l-1,N-l}}^{(l)} = \mathrm{i}\eta$. Here we have also included the equation (v') which implies the behavior of the form factor under Lorentz transformations (v) of (26). The sign factor (ii'_2) follows from (40) and (42) as $(-1)^{n_{l-1}-n_{l+1}}$. This sign does not depend on the specific state of $F_{\underline\alpha}^{\mathcal O}$ but only on the weights of the operator $\mathcal{O}(x)$. This can be seen as follows. We assume that the operator $\mathcal{O}(x)$ transforms under $SU(N)$ as a highest weight representation with highest weight vector

$$w^{\mathcal{O}} = \left(w_1^{\mathcal{O}}, \ldots, w_N^{\mathcal{O}}\right). \tag{49}$$

Then the $SU(N)$ invariance implies that the weight vector of the form factor $F_{\underline{\alpha}}(\underline{\theta})$ as a co-vector is of the form

$$w = w^{\mathcal{O}} + L(1, \ldots, 1), \quad (L = 0, 1, \ldots), \tag{50}$$

because $w = (1, \ldots, 1)$ correspond to the vacuum sector. Then, because of (21),

$$(-1)^{n_{l-1}-n_{l+1}} = (-1)^{w_l+w_{l+1}} = (-1)^{w_l^{\mathcal{O}}+w_{l+1}^{\mathcal{O}}+2L} = (-1)^{w_l^{\mathcal{O}}+w_{l+1}^{\mathcal{O}}}.$$

Similarly, also the phase factor in (ii′) and (iii′) of (48) $\dot{\sigma}(-1)^{n_1+N-1}$ with $\dot{\sigma} = \rho\sigma$ and

$$\rho(-1)^{n_1+N-1} = (-1)^{(1-1/N)(n_0-Q^{\mathcal{O}})+n_1} \tag{51}$$

does not depend on the specific state of $F_{\underline{\alpha}}^{\mathcal{O}}$ but only on the weights of the operator (see (28)).

4. Examples

To illustrate our general results, we present some simple examples of solutions of equations (48) for the p-functions. Details of the following calculations will be published elsewhere [34].

4.1. The energy–momentum tensor

The local operator $\mathcal{O}(x) = T^{\rho\sigma}(x)$ (where $\rho, \sigma = \pm$ denote the light cone components) has charge $Q^T = 0$, is bosonic, transforms as a scalar under $SU(N)$, and has the weight vector of (49) $w^T = (0, \ldots, 0)$. The p-function (as for the sine-Gordon model in [33])

$$p^{T^{\rho\sigma}}(\underline{\theta}, \underline{z}) = N_n^{T^{\rho\sigma}} \sum_{i=1}^{n} \mathrm{e}^{\rho\theta_i} \sum_{i=1}^{m} \mathrm{e}^{\sigma z_i}.$$

solves (48) with

charge	$Q^T = 0$
weight vector	$w^T = (0, \ldots, 0)$
statistics factor	$\sigma^T = 1$
spin	$s^T = 2$

if $N_n^{T^{\rho\sigma}} \big/ N_{n-N}^{T^{\rho\sigma}} = \nu$. The sign factor (51) here is $\rho(-1)^{n_1+N-1} = 1$.

For the $n = N$ particle form factor, there are $n_l = N - l$ integrations in the lth level of the off-shell Bethe ansatz because (see (21) and (50))

$$w = (n - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1}) = (1, 1, \ldots, 1, 1).$$

We calculate the form factor of the particle α and the bound state $(\underline{\beta}) = (\beta_1, \ldots, \beta_{N-1})$ of $N-1$ particles. In each level, all integrations up to one may be performed iteratively using the bound-state relation (iv) (similar as in the proof of theorem 1). The result for the form factor of the particle α and the bound state $(\underline{\beta})$ is written as

$$F_{\alpha(\underline{\beta})}^{T^{\rho\sigma}}(\theta_1, \theta_2) = K_{\alpha(\underline{\beta})}^{T^{\rho\sigma}}(\theta_1, \theta_2) G(\theta_{12})$$

$$K_{\alpha(\underline{\beta})}^{T^{\rho\sigma}}(\theta_1, \theta_2) = N_N^{T^{\rho\sigma}} \left(\mathrm{e}^{\rho\theta_1} + e^{\rho\theta_2}\right) \int_{\mathcal{C}_{\underline{\theta}}} \frac{\mathrm{d}z}{R} \tilde{\phi}(\theta_1 - z)\, \mathrm{e}^{\sigma z} L(\theta_2 - z) \epsilon_{\delta\underline{\gamma}} \tilde{S}_{\alpha\epsilon}^{\delta 1}(\theta_1 - z) \tilde{S}_{(\underline{\beta})1}^{\epsilon(\underline{\gamma})}(\theta_2 - z) \tag{52}$$

where the summation is over $\underline{\gamma}$ and $\delta > 1$ and $G(\theta)$ is the minimal form factor function of two particles of ranks $r = 1$ and $r = N - 1$. The functions $G(\theta)$ and $L(\theta)$ are given by

$$G(\mathrm{i}\pi - \theta)F(\theta)\phi(\theta) = 1,$$
$$L(\theta) = \Gamma\left(\tfrac{1}{2} + \theta/(2\pi\mathrm{i})\right)\Gamma\left(-\tfrac{1}{2} + 1/N - \theta/(2\pi\mathrm{i})\right).$$

The remaining integral in (52) may be performed (similar as in [33]) with the result[5]

$$\langle 0|T^{\rho\sigma}(0)|\theta_1, \theta_2\rangle^{\mathrm{in}}_{\underline{\alpha\beta})} = 4M^2\epsilon_{\underline{\alpha\beta}}\,\mathrm{e}^{\frac{1}{2}(\rho+\sigma)(\theta_1+\theta_2+\mathrm{i}\pi)}\frac{\sinh\frac{1}{2}(\theta_{12} - \mathrm{i}\pi)}{\theta_{12} - \mathrm{i}\pi}G(\theta_{12}).$$

Similar as in [33] one can prove the eigenvalue equation

$$\left(\int \mathrm{d}x T^{\pm 0}(x) - \sum_{i=1}^{n} p_i^{\pm}\right)|\theta_1, \ldots, \theta_n\rangle^{\mathrm{in}}_{\underline{\alpha}} = 0$$

for arbitrary states.

4.2. The fields $\psi_\alpha^{(\pm)}(x)$

Because the Bethe ansatz yields highest weight states, we obtain the matrix elements of the spinor field $\psi(x) = \psi_1^{(+)}(x)$, where we consider here the + component of the spinor. The weight vector is $w^\psi = (1, 0, \ldots, 0)$ and the charge is $Q^\psi = 1$. The p-function for the local operator $\psi(x)$ is (also see [13])

$$p^\psi(\underline{\theta}, \underline{z}) = N_n^\psi \exp\frac{1}{2}\left((1 - 1/N)\sum_{i=1}^{n}\theta_i - \sum_{i=1}^{m} z_i\right).$$

It solves (48) with

charge	$Q^\psi = 1$
weight vector	$w^\psi = (1, \ldots, 0)$
statistics factor	$\sigma^\psi = \mathrm{e}^{\mathrm{i}\pi(1-1/N)}$
spin	$s^\psi = \frac{1}{2}(1 - 1/N)$

if $N_n^\psi / N_{n-N}^\psi = \nu$. The last two formulae are consistent with the proposal of Swieca *et al* [30, 31] that the statistics of the fundamental particles in the chiral $SU(N)$ Gross–Neveu model should be $\sigma = \exp(2\pi\mathrm{i}s)$, where $s = \frac{1}{2}\left(1 - \frac{1}{N}\right)$ is the spin (also see (27)). The sign factor (51) here again is $\rho(-1)^{n_1+N-1} = 1$.

The 1-particle form factor is, for example,

$$\langle 0|\psi(0)|\theta\rangle_\alpha = \delta_{\alpha 1}\,\mathrm{e}^{\frac{1}{2}(1-1/N)\theta}.$$

For the $n = N + 1$ particle form factor, there are again $n_l = N - l$ integrations in the lth level of the off-shell Bethe ansatz and the $SU(N)$ weights are $w = (2, 1, \ldots, 1, 1)$. Similar as above, one obtains the 2-particle and 1-bound state form factor

$$F^\psi_{\alpha\beta(\underline{\gamma})}(\theta_1, \theta_2, \theta_3) = K^\psi_{\alpha\beta(\underline{\gamma})}(\theta_1, \theta_2, \theta_3)F(\theta_{12})G(\theta_{13})G(\theta_{23}),$$

$$K^\psi_{\alpha\beta(\underline{\gamma})} = N^\psi\,\mathrm{e}^{\frac{1}{2}(1-1/N)\sum\theta_i}\int_{\mathcal{C}_{\underline{\theta}}}\frac{\mathrm{d}z}{R}\tilde{\phi}(\theta_1 - z)\tilde{\phi}(\theta_2 - z)L(\theta_3 - z)\,\mathrm{e}^{-\frac{1}{2}z}$$
$$\times\,\epsilon_{\delta\underline{\gamma}}\tilde{S}^{\delta 1}_{\alpha_1\epsilon}(\theta_1 - z)\tilde{S}^{\epsilon 1}_{\alpha_2\zeta}(\theta_2 - z)\tilde{S}^{\zeta(\underline{\gamma})}_{(\underline{\beta})1}(\theta_3 - z).$$

We were not able to perform this integration. In [34] we will discuss the $1/N$ expansion of this exact form factor and compare the result with the $1/N$ expansion for the chiral Gross–Neveu model.

[5] In [41, 39] this result has been obtained using Jackson-type integrals.

4.3. The current $J^{\mu}_{\alpha(\rho)}(x)$

The $SU(N)$ current transforms as the adjoint representation with highest weights $w^J = (2, 1, \ldots, 1, 0)$ and has charge $Q_J = 0$. Again, because the Bethe ansatz yields highest weight states we obtain the matrix elements of the highest weight component

$$J^{\mu}_{\alpha(\rho)} = \delta_{\alpha 1}\epsilon_{(\rho)N} J^{\mu}, \qquad (\rho) = (\rho_1, \ldots, \rho_{N-1})$$
$$J^{\mu} = \epsilon^{\mu\nu}\partial_\nu \varphi$$

with the pseudo-potential $\varphi(x)$. The p-function for the operator $\varphi(x)$

$$p^{\varphi}(\underline{\theta}, \underline{z}, \underline{u}) = N^{\varphi}_n \left(\sum_{i=1}^{n} \exp\theta_i\right)^{-1} \exp\frac{1}{2}\left(\sum_{i=1}^{n_0}\theta_i - \sum_{i=1}^{n_1} z^{(1)}_i - \sum_{i=1}^{n_{N-1}} z^{(N-1)}_i\right)$$

solves (48) with

charge	$Q^{\varphi} = 0$
weight vector	$w^{\varphi} = (2, 1, \ldots, 1, 0)$
statistics factor	$\sigma^{\varphi} = 1$
spin	$s^{\varphi} = 0, s^J = 1$

if $N^{\varphi}_n \big/ N^{\varphi}_{n-N} = \nu$. The sign factor (51) is here $\rho(-1)^{n_1+N-1} = -1$.

We calculate the form factor of the particle α and the bound state $(\lambda) = (\lambda_1, \ldots, \lambda_{N-1})$ of $N-1$ particles with the weight vector $w = (2, 1, \ldots, 1, 0)$. In each level the integrations may be performed iteratively using the bound-state relation (iv), similar as above; however, here no integration remains. The result is

$$F^{\varphi}_{\alpha(\lambda)}(\theta, \omega) = K^{\varphi}_{\alpha(\lambda)}(\theta, \omega) G(\theta - \omega)$$
$$K^{\varphi}_{\alpha(\lambda)}(\theta, \omega) = N^{\varphi}_2 \delta_{\alpha 1}\epsilon_{(\lambda)N} \frac{\mathrm{e}^{\frac{1}{2}(\theta+\omega)}}{\mathrm{e}^{\theta} + \mathrm{e}^{\omega}}.$$

The current form factor therefore is

$$F^{J^{\pm}_{\beta(\rho)}}_{\alpha(\lambda)}(\theta, \omega) = \langle 0 | J^{\pm}_{\beta(\rho)}(0) | \theta, \omega\rangle^{\mathrm{in}}_{\alpha(\lambda)} = \pm N_2 \delta^{\beta}_{\alpha}\delta^{(\rho)}_{(\lambda)} (\mathrm{e}^{\pm\theta} + \mathrm{e}^{\pm\omega}) \frac{\mathrm{e}^{\frac{1}{2}(\theta+\omega)}}{\mathrm{e}^{\theta} + \mathrm{e}^{\omega}} G(\theta - \omega)$$
$$= \delta^{\beta}_{\alpha}\delta^{(\rho)}_{(\lambda)} \bar{v}(\omega)\gamma^{\pm} u(\theta) G(\theta - \omega)/G(\mathrm{i}\pi)$$

as expected.

5. Summary

In this paper we have obtained the form factors for an $SU(N)$ invariant S-matrix. Since the S-matrix also describes backward scattering, the algebraic structure of the form factors was more intricate compared to the $Z(N)$ case [23]. We had to combine the nesting procedure with the techniques of the off-shell Bethe ansatz to capture the vectorial nature of the form factors. The solution was obtained explicitly in terms of multiple contour integrals. We exemplified our general solution for several operators, such as the energy–momentum tensor, the fields and the current. We believe that our new integral representation may shed some light for a better comprehension of complicated objects, such as the correlation functions for the $SU(N)$ invariant Gross–Neveu model.

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Appendix A. The $SU(N)$ bound-state *S*-matrix

The eigenvalue $S_-(\theta)$ has a pole at $\theta = \mathrm{i}\eta = 2\pi\mathrm{i}/N$ which means that there exist bound states of r fundamental particles $\alpha_1 + \cdots + \alpha_r \to (\rho_1 \cdots \rho_r)$(with $\rho_1 < \cdots < \rho_r$) which transform as the anti-symmetric $SU(N)$ tensor representation of rank $r(0 < r < N)$. The masses of the bound states satisfy $m_r = m_{N-r}$ which suggests Swieca's [30, 31, 42] picture that the anti-particle of a particle of rank r is to be identified with the particle of rank $N - r$ (also see [22, 23]).

The general bound-state *S*-matrix formula [32, 33] for the scattering of a bound state with another particle reads as

$$S_{(\rho\sigma)\gamma}^{\gamma'(\rho'\sigma')}(\theta_{(12)3})\Gamma_{\alpha\beta}^{(\rho\sigma)} = \Gamma_{\alpha'\beta'}^{(\rho'\sigma')} S_{\alpha\gamma''}^{\gamma'\alpha'}(\theta_{13}) S_{\beta\gamma}^{\gamma''\beta'}(\theta_{23})|_{\theta_{12}=\mathrm{i}\eta} \tag{A.1}$$

where $\theta_{(12)}$ is the bound-state rapidity and η is the bound-state fusion angle. The bound-state fusion intertwiner $\Gamma_{\alpha\beta}^{(\rho\sigma)}$ is defined by

$$\mathrm{i} \operatorname*{Res}_{\theta=i\eta} S_{\alpha\beta}^{\beta'\alpha'}(\theta) = \sum_{\rho<\sigma} \Gamma_{(\rho\sigma)}^{\beta'\alpha'} \Gamma_{\alpha\beta}^{(\rho\sigma)}. \tag{A.2}$$

With a convenient choice of an undetermined phase factor, one obtains

$$\Gamma_{\alpha\beta}^{(\rho\sigma)} = \Gamma_{(\rho\sigma)}^{\beta\alpha} = \mathrm{i}\sqrt{\mathrm{i}a(\mathrm{i}\eta)\mathrm{i}\eta}\left(\delta_\alpha^\rho \delta_\beta^\sigma - \delta_\alpha^\sigma \delta_\beta^\rho\right). \tag{A.3}$$

Iterating the general bound-state formula $N-1$ times, one obtains for the scattering of a bound state $(\rho) = (\rho_1\rho_2 \cdots \rho_{N-1})$ with another particle δ:

$$S_{(\rho)\delta}^{\delta'(\sigma)}(\theta)\Gamma_{\underline{\alpha}}^{(\rho)} = \Gamma_{\underline{\alpha}'}^{(\sigma)} S_{\alpha_1\delta_1}^{\delta'\alpha_1'}(\theta + \mathrm{i}\pi - \mathrm{i}\eta) \cdots S_{\alpha_{N-1}\delta}^{\delta_{N-2}\alpha_{N-1}'}(\theta - \mathrm{i}\pi + \mathrm{i}\eta)$$

with the total bound-state fusion intertwiner

$$\Gamma_{\underline{\alpha}}^{(\rho)} = \Gamma_{\alpha_1\alpha_2\cdots\alpha_{N-1}}^{(\rho_1\rho_2\cdots\rho_{N-1})} = \Gamma_{(\rho_1\rho_2\cdots\rho_{N-2})\alpha_{N-1}}^{(\rho_1\rho_2\cdots\rho_{N-1})} \cdots \Gamma_{(\rho_1\rho_2)\alpha_3}^{(\rho_1\rho_2\rho_3)} \Gamma_{\alpha_1\alpha_2}^{(\rho_1\rho_2)}.$$

Taking special cases for the external particles, we obtain

$$S_{(1,2\cdots,N-1)N}^{N(1,2\cdots,N-1)}(\theta) = b(\theta + \mathrm{i}\pi - \mathrm{i}\eta) \cdots b(\theta - \mathrm{i}\pi + \mathrm{i}\eta) = (-1)^{N-1} a(\pi\mathrm{i} - \theta)$$

$$S_{(1,2\cdots,N-1)N-1}^{N-1(1,2\cdots,N-1)}(\theta) = b(\theta + \mathrm{i}\pi - \mathrm{i}\eta) \cdots a(\theta - \mathrm{i}\pi + \mathrm{i}\eta) = (-1)^{N-1} b(\pi\mathrm{i} - \theta)$$

$$S_{(2,3\cdots,N)1}^{N(1,2\cdots,N-1)}(\theta) = (-1)^N b(\theta + \mathrm{i}\pi - \mathrm{i}\eta) \cdots c(\theta - \mathrm{i}\pi + \mathrm{i}\eta) = c(\pi\mathrm{i} - \theta).$$

These results may be interpreted as an unusual crossing relation:

$$\begin{aligned} \mathbf{C}_{(\beta)\beta}\Gamma_{\underline{\beta}}^{(\beta)}(S(\theta_1)\cdots S(\theta_{N-1}))_{\underline{\alpha}\delta}^{\delta'\underline{\beta}} &= \mathbf{C}_{(\beta)\beta} S_{(\alpha)\delta}^{\delta'(\beta)}(\theta)\Gamma_{\underline{\alpha}}^{(\alpha)} \\ &= (-1)^{N-1}\mathbf{C}_{(\alpha)\gamma}\Gamma_{\underline{\alpha}}^{(\alpha)} S_{\delta\beta}^{\gamma\delta'}(\mathrm{i}\pi - \theta) \end{aligned} \tag{A.4}$$

with $\theta_j = \theta + \mathrm{i}\pi - j\mathrm{i}\eta$ and the charge conjugation matrix $\mathbf{C}_{(\beta)\beta}$ defined by (4). The total bound-state fusion intertwiner satisfies

$$\mathbf{C}_{(\beta_1\cdots\beta_{N-1})\gamma}\Gamma_{\alpha_1\cdots\alpha_{N-1}}^{(\beta_1\cdots\beta_{N-1})} = \epsilon_{\alpha_1\cdots\alpha_{N-1}\gamma}\Gamma, \Gamma = \mathrm{i}\sqrt{\frac{1}{2\pi}N\Gamma^N(1-1/N)}. \tag{A.5}$$

We consider the total N-particle S-matrix (consisting of $N(N-1)/2$ factors) in terms of $\tilde S$:

$$\tilde S_{12\cdots N} = (\tilde S_{12}\tilde S_{13}\cdots\tilde S_{1N})(\tilde S_{23}\cdots\tilde S_{2N})\cdots\tilde S_{N-1N} \tag{A.6}$$

in the limit $\theta_{jj+1}\to \mathrm{i}\eta (j=1,\ldots N-1)$. It behaves as

$$\tilde S^{\beta_N\cdots\beta_2\beta_1}_{\alpha_1\alpha_2\cdots\alpha_N} \approx (N-1)!\frac{\mathrm{i}\eta}{\theta_{12}-\mathrm{i}\eta}\cdots\frac{\mathrm{i}\eta}{\theta_{N-1N}-\mathrm{i}\eta}\epsilon^{\beta_1\cdots\beta_N}\epsilon_{\alpha_1\cdots\alpha_N}. \tag{A.7}$$

The algebraic structure of this relation follows because one can use the Yang–Baxter equations to shift in (A.6) any factor $\tilde S_{ii+1}\sim(1-\mathbf{P})_{ii+1}$ to the right or the left. Therefore, the expression is totally anti-symmetric with respect to α_i and β_i. The factor follows from (6).

The normalization constant c in (9) is calculated from (11) and (10) as

$$c = \Gamma^{-2(1-1/N)}\left(\frac12-\frac1{2N}\right)\exp\left(-\int_0^\infty \mathrm{e}^{\frac1N t}\left(\frac{\sinh\left(1-\frac1N\right)t}{t\sinh^2 t}-\frac{\left(1-\frac1N\right)}{t\sinh t}\right)\mathrm{d}t\right)>0.$$

Appendix B. Proof of theorem 1

Proof. Property (i) in the form of (30) follows directly from (i′) of (40), the Yang–Baxter equations and the action of the S-matrix on the pseudo-ground state Ω

$$\begin{aligned}\Omega_{\ldots ji\ldots}\tilde C_{\ldots ji\ldots}(\cdots\theta_j,\theta_i\cdots)\tilde S_{ij}(\theta_{ij}) &= \Omega_{\ldots ji\ldots}\tilde S_{ij}(\theta_{ij})\tilde C_{\ldots ij\ldots}(\cdots\theta_i,\theta_j\cdots)\\ &= \Omega_{\ldots ij\ldots}\tilde C_{\ldots ij\ldots}(\cdots\theta_i,\theta_j\cdots)\end{aligned}$$

because $\tilde S^{11}_{11}(\theta) = S^{11}_{11}(\theta)/a(\theta) = 1$ and $F(\theta)=F(-\theta)a(\theta)$.

Using (i) and (5) the property (ii) in the form of (31) may be rewritten as a matrix difference equation [39, 41, 43]:

$$K_{1\cdots n}(\underline\theta)Q_{1\cdots n}(\underline\theta,i) = (-1)^{N-1}K_{1\cdots n}(\underline\theta')\dot\sigma_1 \tag{B.1}$$

where $\underline\theta' = (\theta_1,\ldots,\theta_i+2\pi\mathrm{i},\ldots,\theta_n)$ and $\dot\sigma$ is a statistics factor. The matrix $Q(\underline\theta,i)$ is the trace with respect to the auxiliary space

$$Q_{1\cdots n}(\underline\theta,i) = \mathrm{tr}_0\tilde T_{Q,1\cdots n,0}(\underline\theta,i) \tag{B.2}$$

of a modified monodromy matrix

$$\tilde T_{Q,1\cdots n,0}(\underline\theta,i) = \tilde S_{10}(\theta_1-\theta_i')\cdots\mathbf{P}_{i0}\cdots\tilde S_{n0}(\theta_n-\theta_i)$$

where $\mathbf{P}=\tilde S(0)$ is the permutation matrix. In (B.2) we use the rule that the rapidity of a line changes by $2\pi\mathrm{i}$ if the line bends by 360° in the positive sense. Because of (i) it is sufficient to proof the matrix difference equation (B.1) for $i=1$, i.e. for $Q(\underline\theta)=Q(\underline\theta,1)$.

In the following, we will suppress the indices $1\cdots n$. The Yang–Baxter relations (14) imply the typical commutation rules for the matrices $\tilde A,\tilde C,\tilde D$ defined in (14):

$$\begin{aligned}\tilde C^\beta(\underline\theta,z)\tilde A(\underline\theta,\theta) &= \frac{1}{\tilde b(\theta-z)}\tilde A(\underline\theta,\theta)\tilde C^\beta(\underline\theta,z)-\frac{\tilde c(\theta-z)}{\tilde b(\theta-z)}\tilde A(\underline\theta,z)\tilde C^\beta(\underline\theta,\theta)\\ \tilde C^\beta(\underline\theta,z)\tilde D^{\gamma'}_\gamma(\underline\theta,\theta) &= \frac{1}{\tilde b(z-\theta)}\tilde S^{\gamma'\beta}_{\beta'\gamma''}(z-\theta)\tilde D^{\gamma''}_\gamma(\underline\theta,\theta)\tilde C^{\beta'}(\underline\theta,z)\\ &\quad-\frac{\tilde c(z-\theta)}{\tilde b(z-\theta)}\tilde D^\beta_\gamma(\underline\theta,z)\tilde C^{\gamma'}(\underline\theta,\theta)\end{aligned} \tag{B.3}$$

where $\beta,\beta',\gamma,\gamma',\gamma''\in\{2,\ldots,N\}$. In addition, there are the Zapletal commutation rules [39, 41, 43] where the matrices A_Q, C_Q, D_Q defined by

$$\tilde T_Q(\underline\theta) = \begin{pmatrix}\tilde A_Q(\underline\theta) & \tilde B_Q^\beta(\underline\theta)\\ \tilde C_Q^\beta(\underline\theta) & \tilde D_{Q\beta}^{\ \beta'}(\underline\theta)\end{pmatrix}$$

are also involved [43]:

$$\tilde{C}^{\beta}(\underline{\theta}, z)\tilde{A}_Q(\underline{\theta}) = \frac{1}{\tilde{b}(\theta_1' - z)}\tilde{A}_Q(\underline{\theta})\tilde{C}^{\beta}(\underline{\theta}', z) - \frac{\tilde{c}(\theta_1' - z)}{\tilde{b}(\theta_1' - z)}\tilde{A}(\underline{\theta}, z)\tilde{C}_Q^{\beta}(\underline{\theta}) \tag{B.4}$$

$$\begin{aligned}\tilde{C}^{\beta}(\underline{\theta}, z)\tilde{D}_{Q}{}^{\gamma'}_{\gamma}(\underline{\theta}) &= \frac{1}{\tilde{b}(z - \theta_1')}\tilde{S}^{\gamma'\beta}_{\beta'\gamma''}(z - \theta_1')\tilde{D}_{Q}{}^{\gamma''}_{\gamma}(\underline{\theta})\tilde{C}^{\beta'}(\underline{\theta}', z)\\ &\quad - \frac{\tilde{c}(z - \theta_1)}{\tilde{b}(z - \theta_1)}\tilde{D}^{\beta}_{\gamma}(\underline{\theta}, z)\tilde{C}^{\gamma'}_{Q}(\underline{\theta}).\end{aligned} \tag{B.5}$$

Note that we assign to the auxiliary space of $\tilde{T}_Q(\underline{\theta})$ corresponding to the horizontal line the spectral parameter θ_1 on the right-hand side and $\theta_1' = \theta_1 + 2\pi \mathrm{i}$ on the left-hand side.

We are now going to prove (B.1) in the form

$$K(\underline{\theta})Q(\underline{\theta}) = K(\underline{\theta})\left(\tilde{A}_Q(\underline{\theta}) + \sum_{\beta=2}^{N}\tilde{D}_{Q}{}^{\beta}_{\beta}(\underline{\theta})\right) = \dot{\sigma}(-1)^{N-1}K(\underline{\theta}') \tag{B.6}$$

where $K(\underline{\theta})$ is a co-vector-valued function as given by (15) and the Bethe ansatz state (18) and (19). Using a short notation, we write

$$K(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} \mathrm{d}\underline{z}\,\tilde{h}(\underline{\theta}, \underline{z})p(\underline{\theta}, \underline{z})\tilde{\Psi}(\underline{\theta}, \underline{z})$$

(with $\int_{\mathcal{C}_{\underline{\theta}}} \mathrm{d}\underline{z} = \frac{N_n}{m!}\int_{\mathcal{C}_{\underline{\theta}}}\frac{\mathrm{d}z_1}{R}\cdots\int_{\mathcal{C}_{\underline{\theta}}}\frac{\mathrm{d}z_m}{R}$). To analyze the left-hand side of (B.6), we proceed as follows: we apply the trace of $\tilde{T}_Q$ to the co-vector $\tilde{\Psi}(\underline{\theta}, \underline{z})$.

In the contribution from $\tilde{A}_Q(\underline{\theta})$ which means $\gamma' = \gamma = 1$, one may use Yang–Baxter relations to observe that only the amplitudes $\tilde{S}^{11}_{11}(\theta_1 - z_j) = 1$ appear in the S-matrices $\tilde{S}(\theta_1 - z_j)$ which are constituents of the C-operators. Therefore we may shift all z_j-integration contours $\mathcal{C}_{\underline{\theta}}$ to $\mathcal{C}_{\underline{\theta}'}$ without changing the values of the integrals, because there are no singularities inside $\mathcal{C}_{\underline{\theta}} \cup -\mathcal{C}_{\underline{\theta}'}$ (cf figure 1).

We now proceed as usual in the algebraic Bethe ansatz and push $\tilde{A}_Q(\underline{\theta})$ and $\tilde{D}_Q(\underline{\theta})$ through all the C-operators using the commutation rules (B.4) and (B.5) and obtain

$$\begin{aligned}\tilde{C}^{\beta_m}(\underline{\theta}, z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta}, z_1)\tilde{A}_Q(\underline{\theta}) &= \prod_{j=1}^{m}\frac{1}{\tilde{b}(\theta_1' - z_j)}\tilde{A}_Q(\underline{\theta})\\ &\quad\times\tilde{C}^{\beta_m}(\underline{\theta}', z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta}', z_1) + \sum uw_A,\\ \tilde{C}^{\beta_m}(\underline{\theta}, z_m)\cdots\tilde{C}^{\beta_1}(\underline{\theta}, z_1)\tilde{D}_{Q}{}^{\beta'}_{\beta}(\underline{\theta}) &= \prod_{j=1}^{m}\frac{1}{\tilde{b}(z_j - \theta_1')}\tilde{T}^{(1)}{}^{\beta'\underline{\beta}}_{\underline{\beta}'\beta''}(\underline{z}, \theta_1')\tilde{D}_{Q}{}^{\beta''}_{\beta}(\underline{\theta})\\ &\quad\times\tilde{C}^{\beta_m'}(\underline{\theta}, z_m)\cdots\tilde{C}^{\beta_1'}(\underline{\theta}, z_1) + \sum uw_D.\end{aligned}$$

The 'wanted terms' written out explicitly originate from the first term in the commutation rules (B.4); all other contributions yield the so-called 'unwanted terms'. The next level monodromy matrix is

$$\tilde{T}^{(1)}{}^{\beta'\underline{\beta}}_{\underline{\beta}'\beta}(\underline{z}, \theta) = (\tilde{S}_{10}(z_1 - \theta)\cdots\tilde{S}_{m0}(z_m - \theta))^{\beta'\underline{\beta}}_{\underline{\beta}'\beta}$$

where β's and also the internal summation indices take the values $2, \ldots, N$. If we insert these equations into the representation (15) of $K(\underline{\theta})$, we first find that the wanted contribution from $\tilde{A}_Q$ already gives the result we are looking for. Secondly, the wanted contribution from $\tilde{D}_Q$ applied to Ω gives zero. Thirdly, the unwanted contributions from $\tilde{A}_Q$ and $\tilde{D}_Q$ cancel after integration over z_j. These three facts follow from the 'shift relations' of the ϕ-function (12) and of the higher level L-function (ii)$^{(1)}$ of (37).

The proof of (iii) is similar to that for the $Z(N)$ model in [23]. We use the short-hand notations

$$\begin{aligned}
&\underline{\theta} = (\theta_1, \ldots, \theta_n), && \underline{\hat{\theta}} = (\theta_1, \ldots, \theta_N), && \underline{\check{\theta}} = (\theta_{N+1}, \ldots, \theta_n),\\
&\underline{\alpha} = (\alpha_1, \ldots, \alpha_n), && \underline{\hat{\alpha}} = (\alpha_1, \ldots, \alpha_N), && \underline{\check{\alpha}} = (\alpha_{N+1}, \ldots, \alpha_n),\\
&\underline{z} = (z_1, \ldots, z_m), && \underline{\hat{z}} = (z_1, \ldots, z_{N-1}), && \underline{\check{z}} = (z_N, \ldots, z_m),\\
&\underline{\beta} = (\beta_1, \ldots, \beta_m), && \underline{\hat{\beta}} = (\beta_1, \ldots, \beta_{N-1}), && \underline{\check{\beta}} = (\beta_N, \ldots, \beta_m).
\end{aligned}$$

We prove the form factor equation in the form of (32):

$$\begin{aligned}
\operatorname*{Res}_{\theta_{N-1N}=i\eta} \cdots \operatorname*{Res}_{\theta_{12}=i\eta} K^{\mathcal{O}}_{1\cdots N}(\underline{\theta}) &= c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde{\phi}(\theta_{ij}) \varepsilon_{1\cdots N}\\
&\quad\times K^{\mathcal{O}}_{N+1\cdots n}(\theta_{N+1}, \ldots, \theta_n)(\mathbf{1} - \dot{\sigma}^{\mathcal{O}}_N S_{Nn} \cdots S_{NN+1})
\end{aligned} \tag{B.7}$$

for the K-function. The residues of $K_{1\cdots n}(\underline{\theta})$ consist of three terms

$$\operatorname{Res}_{\theta_{N-1N}=\mathrm{i}\eta} \cdots \operatorname{Res}_{\theta_{12}=\mathrm{i}\eta} K_{1\cdots n}(\underline{\theta}) = R^{(1)}_{1\cdots n} + R^{(2)}_{1\cdots n} + R^{(3)}_{1\cdots n}$$

because $N-1$ of the z integration contours will be 'pinched' at three points. Again due to symmetry it is sufficient to determine the contribution from the $z_1, \ldots, z_{N-1}$-integrations and multiply the result by $m \cdots (m-N+2)$. The pinching points are

(1) $z_1 = \theta_2, \ldots, z_{N-1} = \theta_N$,
(2) $z_1 = \theta_1, \ldots, z_{N-1} = \theta_{N-1}$,
(3) $z_1 = \theta_2 - i\eta, \ldots, z_{N-1} = \theta_N - \mathrm{i}\eta$.

The contribution of (1) is given by $N-1$ integrations along small circles around $z_1 = \theta_2, z_2 = \theta_3, \ldots, z_{N-1} = \theta_N$ (see figure 1). The S-matrices $\tilde{S}(\theta_2 - z_1), \ldots, \tilde{S}(\theta_N - z_{N-1})$ yield the permutation operator $\tilde{S}(0) = \mathbf{P}$. Therefore for $\theta_{12}, \ldots, \theta_{N-2N-1}, \theta_{N-1N} \to \mathrm{i}\eta$ and $z_1 = \theta_2, z_2 = \theta_3, \ldots, z_{N-1} = \theta_N$

$$\begin{aligned}
\tilde{\Phi}^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) &\to (\Omega \tilde{C}^{\beta_m}(\underline{\theta}, z_m) \cdots \tilde{C}^{\beta_N}(\underline{\theta}, z_N) \tilde{C}^{\beta_{N-1}}(\underline{\theta}, \theta_N) \cdots \tilde{C}^{\beta_1}(\underline{\theta}, \theta_2))_{\underline{\alpha}}\\
&= \prod_{j=N}^{m} \tilde{b}(\theta_1 - z_j)(\tilde{S}_{1N} \cdots \tilde{S}_{12})^{\underline{\hat{\beta}},1}_{\underline{\hat{\alpha}}} (\Omega \tilde{C}^{\beta_m}(\underline{\check{\theta}}, z_m) \cdots \tilde{C}^{\beta_N}(\underline{\check{\theta}}, z_N))_{\underline{\check{\alpha}}}.
\end{aligned} \tag{B.8}$$

It has been used that due to the $SU(N)$ ice rule, only the amplitude $b(\cdot)$ (because all $\beta > 1$) contributes to the S-matrices $S(\theta_1 - z_j)(j \geqslant N)$ and $a(\cdot)$ to the S-matrices $S(\theta_2 - z_j), \ldots, S(\theta_N - z_j)(j \geqslant N), S(\theta_i - z_1), \ldots, S(\theta_i - z_{N-2})(i \geqslant N)$ after having applied Yang–Baxter relations. One observes that the product of S-matrices in (38) together with that in (B.8) yields the total $N-1$ particle S-matrix:

$$\tilde{S}^{N\cdots 2}_{\underline{\hat{\beta}}}(\underline{\hat{z}})(\tilde{S}_{1N} \cdots \tilde{S}_{12})^{\underline{\hat{\beta}},1}_{\underline{\hat{\alpha}}} = \tilde{S}^{N\cdots 21}_{\underline{\hat{\alpha}}}(\underline{\hat{\theta}})$$

for which the residue formula (A.7) applies

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}\eta} \cdots \operatorname*{Res}_{\theta_{N-1N}=i\eta} \tilde{S}^{N\cdots 21}_{\alpha_1\cdots\alpha_N} = (N-1)!(\mathrm{i}\eta)^{N-1} \epsilon^{1\cdots N} \epsilon_{\alpha_1\cdots\alpha_N}.$$

We combine (B.8) with the function $L_{\underline{\beta}}(\underline{z})$ with the property (38) for $(l=1)$ and the scalar functions $\tilde{h}$ and p and after having performed the remaining z_j-integrations we obtain

$$R^{(1)}_{1\cdots n} = c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde\phi(\theta_{ij}) \varepsilon_{1\ldots N}(\underline{\hat\theta}) K_{N+1\cdots n}(\underline{\check\theta})$$

$$c_0 = c_1 \nu (N-1)!(\mathrm{i}\eta)^{N-1}(-1)^{m-N+1} \prod_{j=1}^{N-1} \tilde\phi(j\mathrm{i}\eta).$$

We have used equations (12), (17), relation (iii') of (40) for the p-function, relation (41) for the constant ν and the recursion relation (43) for the constants c_l with solution (C.2).

The remaining contribution to (B.7) is due to R_2 and R_3:

$$R^{(2)}_{1\cdots n} + R^{(3)}_{1\cdots n} = -c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde\phi(\theta_{ij}) \varepsilon_{1\ldots N} K_{N+1\cdots n}(\underline{\check\theta}) \dot\sigma_N S_{Nn} \cdots S_{NN+1}.$$

It is convenient to shift the particle with momentum θ_N to the right by applying S-matrices and write the claim as

$$\left(R^{(2)}_{1\cdots n} + R^{(3)}_{1\cdots n}\right) S_{N+1N} \cdots S_{nN} + c_0 \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde\phi(\theta_{ij}) \varepsilon_{1\ldots N} K_{N+1\cdots n}(\underline{\check\theta}) \dot\sigma_N = 0 \tag{B.9}$$

where the components of this co-vector are now denoted by $v_{\alpha_1\cdots\alpha_{N-1}\alpha_{N+1}\cdots\alpha_n\alpha_N}$. Note that because of (i) (see (30)),

$$K_{1\cdots n}(\underline{\theta}) S_{N+1N} \cdots S_{nN} = \prod_{i=N+1}^{n} a(\theta_{iN}) K_{1\ldots N-1N+1\cdots nN}(\theta_1, \ldots, \theta_{N-1}, \theta_{N+1}, \ldots, \theta_n, \theta_N).$$

Because the Bethe ansatz states are of highest weight (see [36]), it is sufficient to prove (B.9) only for $\alpha_N = 1$. The contribution of $R^{(2)}_{1\cdots n}$ is given by the $z_1, \ldots, z_{N-1}$-integrations along the small circles around $z_1 = \theta_1, \ldots, z_{N-1} = \theta_{N-1}$ (again see figure 1). Now $\tilde S(\theta_1 - z_1)_1, \ldots, \tilde S(\theta_{N-1} - z_{N-1})$ yield permutation operators $\mathbf{P}$, and the co-vector part of this contribution for $\alpha_N = 1$ is

$$\begin{aligned}(\Omega \tilde C^{\beta_m}(\underline{\theta}, z_m) \cdots \tilde C^{\beta_{N-1}}(\underline{\theta}, \theta_{N-1}) \cdots \tilde C^{\beta_1}(\underline{\theta}, \theta_1) P_N(1))_{\alpha_1\cdots\alpha_{N-1}\alpha_{N+1}\cdots\alpha_n\alpha_N} \\ = \delta^{\beta_1}_{\alpha_1} \cdots \delta^{\beta_{N-1}}_{\alpha_{N-1}} \left(\Omega \tilde C^{\beta_m}(\underline{\check\theta}, z_m) \cdots \tilde C^{\beta_N}(\underline{\check\theta}, z_N)\right)_{\underline{\check\alpha}} \delta^1_{\alpha_N}\end{aligned} \tag{B.10}$$

where $P_N(1)$ projects onto the components with $\alpha_N = 1$. We have used the fact that because of the $SU(N)$ ice rule, the amplitude $a(\cdot)$ only contributes to the S-matrices $S(\theta_1 - z_j)$, $S(\theta_2 - z_j)$, $S(\theta_N - z_j)$, $S(\theta_i - z_1)$ after having applied the Yang–Baxter relations. We use $\tilde\phi(\theta) = -\tilde b(\theta + 2\pi\mathrm{i})\tilde\phi(\theta + 2\pi\mathrm{i})$ to replace for $i = 1, \ldots, N-1$ and $\beta \neq 1$

$$\tilde\phi(\theta_{Ni}) = \tilde S^{11}_{11}(\theta_{Ni})\tilde\phi(\theta_{Ni}) = -\tilde S^{\beta 1}_{1\beta}(\theta_{Ni} + 2\pi\mathrm{i})\tilde\phi(\theta_{Ni} + 2\pi\mathrm{i})$$

therefore using again (A.7)

$$\begin{aligned}\operatorname*{Res}_{\theta_{N-1N}=\mathrm{i}\eta} \cdots \operatorname*{Res}_{\theta_{12}=\mathrm{i}\eta} \tilde\phi(\theta_{N1}) \cdots \tilde\phi(\theta_{NN-1}) \tilde S^{N\cdots 2}_{\alpha_1\cdots\alpha_{N-1}}(\theta_1, \ldots, \theta_{N-1}) \\ = -\tilde\phi(\mathrm{i}\eta) \cdots \tilde\phi((N-1)\mathrm{i}\eta)(N-1)!(\mathrm{i}\eta)^{N-1} \epsilon_{\alpha_1\cdots\alpha_{N-1}1}.\end{aligned}$$

We combine (B.10) with the function $L_{\underline{\beta}}(\underline{z})$ with the property (38) (for $l = 1$) and the scalar functions $\tilde h$ and p and after having performed the remaining z_j-integrations, we obtain

$$R^{(2)}_{1\cdots n} S_{N+1N} \cdots S_{nN} P_N(1) = -c_0 \dot\sigma \prod_{i=N+1}^{n} \prod_{j=2}^{N} \tilde\phi(\theta_{ij}) \varepsilon_{1\ldots N} K_{N+1\cdots n}(\underline{\check\theta}) P_N(1). \tag{B.11}$$

Again we have used equations (12), (17), relation (iii') of (40) for the p-function, relation (41) for the constant ν and the recursion relation (43) for the constants c_l with solution (C.2).

The contribution of pinching (3) is given by the $z_1, \ldots, z_{N-1}$-integrations along the small circles around $z_1 = \theta_2 - i\eta, \ldots, z_{N-2} = \theta_{N-1} - i\eta, z_{N-1} = \theta_N - i\eta$, (again see figure 1). In $\tilde{C}^{\beta_{N-1}}_{\underline{\tilde{\alpha}}}(\underline{\theta}, z_{N-1})$ the S-matrix $\tilde{S}^{\beta'_{N-1}\alpha''_N}_{\alpha'_N 1}(\theta_N - z_{N-1})$ yields $\Gamma^{\beta'_{N-1}\alpha''_N}_{(\rho\sigma)}\Gamma^{(\rho\sigma)}_{\alpha'_N 1}$ by applying $\mathrm{Res}_{z_{N-1}=\theta_N - i\eta}$. (For $\alpha_N = 1$ this vanishes because $\Gamma^{(\rho\sigma)}_{\alpha\beta}$ is anti-symmetric with respect to α, β, the $SU(N)$ ice rule also implies that $\alpha'_N = 1$.) Therefore (B.9) is proved for $\alpha_N = 1$ and also because the left-hand side belongs to the highest weight state in general. □

Appendix C. Proof of lemma 2

Proof. We set $n_l = m, n_{l+1} = k$. For the higher level functions $L^{(l)}_{\underline{\beta}}(\underline{z})$, one may verify equations (i)$^{(l)}$ and (ii)$^{(l)}$ quite analogously to the corresponding ones for the main theorem. We prove (iii)$^{(l)}$ by induction and assume

$$L^{(l+1)}_{\underline{\gamma}}(\underline{u}) \approx c_{l+1}\tilde{S}^{N\cdots l+2}_{\underline{\hat{\gamma}}}(\underline{\hat{u}}) \prod_{i=N-l}^{k}\prod_{j=2}^{N-l-1}\tilde{\phi}(u_{ij})L^{(l+1)}_{\underline{\check{\gamma}}}(\underline{\check{u}}) \tag{C.1}$$

for $u_{12}, \ldots, u_{N-l-2,N-2} \to i\eta$. In the integral representation (34) of $L^{(l)}_{\underline{\beta}}(\underline{z})$, there are pinchings at $u_1 = z_2, \ldots, u_{N-l-1} = z_{N-l}$ if $z_{12}, \ldots, z_{N-l-1,N-l} \to i\eta$. Therefore in $\tilde{\Psi}^{(l)}_{\underline{\beta}}(\underline{z}, \underline{u})$ the S-matrices $\tilde{S}(z_2 - u_1), \ldots, \tilde{S}(z_{N-l} - u_{N-l-1})$ yield the permutation operator $\mathbf{P}$, and we have to consider

$$\begin{aligned}
\tilde{\Phi}^{(l)\underline{\gamma}}_{\underline{\beta}}(\underline{z}, \underline{u}) &= \left(\Omega^{(l)}\tilde{C}^{(l)\gamma_k}(\underline{z}, u_k)\cdots\tilde{C}^{(l)\gamma_{N-l-1}}(\underline{z}, z_{N-l})\cdots\tilde{C}^{(l)\gamma_1}(\underline{z}, z_2)\right)_{\underline{\beta}} \\
&= \left(\tilde{S}_{1N-l}(z_{1N-l})\cdots\tilde{S}_{12}(z_{12})\right)^{\gamma_1\cdots\gamma_{N-l-1}l+1}_{\beta_1\cdots\beta_{N-l}}\prod_{j=N-l}^{k}\tilde{b}(z_1 - u_j)\tilde{\Phi}^{(l)\underline{\check{\gamma}}}_{\underline{\check{\beta}}}(\underline{\check{z}}, \underline{\check{u}})
\end{aligned}$$

where $\underline{\check{\beta}} = (\beta_{N-l+1}, \ldots, \beta_m), \underline{\check{z}} = (z_{N-l+1}, \ldots, z_m), \underline{\check{u}} = (u_{N-l}, \ldots, u_k)$. We may write for $\underline{\hat{u}} = (u_1, \ldots, u_{N-l-1}) = (z_2, \ldots, z_{N-l})$

$$\tilde{S}^{N\cdots l+2}_{\underline{\hat{\gamma}}}(\underline{\hat{u}})\left(\tilde{S}_{1N-l}(z_{1N-l})\cdots\tilde{S}_{12}(z_{12})\right)^{\gamma_1\cdots\gamma_{N-l-1}l+1}_{\beta_1\cdots\beta_{N-l}} = \tilde{S}^{N\cdots l+1}_{\underline{\hat{\beta}}}(\underline{\hat{z}})$$

with the notation $\underline{\hat{z}} = (z_1, \ldots, z_{N-l})$. Therefore using the assumption (C.1) we obtain when $z_{12}, \ldots, z_{N-l-1,N-l} \to i\eta$

$$\begin{aligned}
L^{(l)}_{\underline{\beta}}(\underline{z}) &\approx \frac{1}{\check{k}!}\oint_{z_2}\frac{du_1}{R}\cdots\oint_{z_{N-l}}\frac{du_{N-l-1}}{R}\int_{\mathcal{C}_{\underline{z}}}\frac{du_{N-l}}{R}\cdots\int_{\mathcal{C}_{\underline{z}}}\frac{du_k}{R}\tilde{h}(\underline{z}, \underline{u}) \\
&\quad\times L^{(l+1)}_{\underline{\gamma}}(\underline{u})\tilde{\Phi}^{(l)\underline{\gamma}}_{\underline{\beta}}(\underline{z}, \underline{u}) = c_l\tilde{S}^{N\cdots l+1}_{\underline{\hat{\beta}}}(\underline{\hat{z}})\prod_{i=N-l+1}^{m}\prod_{j=2}^{N-l}\tilde{\phi}(z_{ij})L^{(l)}_{\underline{\check{\beta}}}(\underline{\check{z}})
\end{aligned}$$

where $\check{k} = k - N + l + 1$. The following formulae have been used:

$$\tilde{b}(z_1 - u_j)\prod_{i=3}^{N-l}\tilde{\phi}(u_j - z_i)\prod_{i=1}^{N-l}\tilde{\phi}(z_i - u_j)\prod_{i=2}^{N-l}\tau(z_i - u_j) = -1,$$

relation (iii$'$)$^{(l)}$ of (42) for the p-function and the recursion relation

$$c_l = c_{l+1}(-1)^{n_{l+1}-N+l+1}\prod_{j=1}^{N-l-1}\tilde{\phi}(ji\eta).$$

The solution of this recursion relation with $c_{N-1} = 1$ is

$$c_1 = \tilde{\phi}^{N-2}(\mathrm{i}\eta)\tilde{\phi}^{N-3}(2\mathrm{i}\eta)\cdots\tilde{\phi}((N-2)\mathrm{i}\eta)\prod_{l=1}^{N-2}(-1)^{n_{l+1}-N+l+1}. \tag{C.2}$$

$\square$

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